A dynamical approach to geometry

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A short journey to a dynamical approach of geometry

The classical study of geometry begins with the Greeks. They introduced the notion of geometrical object characterizing them by what we call now **invariants**, i.e. quantities which do not change under a family of given transformations. For example, triangle has always three vertices and edges. This period is resumed in the famous book of Euclide "Elements". Many works have then been devoted to introduce new invariants in particular invariance under **symmetries group**.
However, all these results are related to a particular description of the object involving in an ambient space. The underlying point of view is then called **extrinsic**.

Many problems, especially in Physics, call for the development of an **intrinsic** geometry, meaning that all the properties of the object must be described by quantities that can be computed only using information on the object. A typical question from this point of view is: how to know that we are on a sphere by walking on it ?

Euclide
(325BC - 265BC)

C.F.Gauss
(1777-1855)

A fundamental breakthrough is made by Gauss in ∼1810 which open the field of what is now called **differential geometry** and **Riemannian geometry**. Despite its success many natural problems remain unsolved in the classical framework of differential geometry.

A new point of view, taking its sources in the study of solutions of polynomial equations, rise from the possibility to replace points by algebras and classical geometrical approach by computations on these algebras. This point of view is part of what is called **algebraic geometry** culminating with the work of A.Grothendieck. The algebraic approach to geometry is very powerful and has lead to many deep and hard results which are balanced by a lost of an intuitive understanding of some of its concepts.
Algebraic geometry was generalized to a bigger class of object by A. Connes in its **noncommutative geometry**.

A.Grothendieck
(1928 - 2014)

A.Connes
(1947-...)

In the following, we introduce a new point of view called **dynamical geometry** which gives a **new understanding** to some **algebraic constructions** as well as new results. In particular, this point of view makes a **dictionary** between geometrical properties of an object, some algebraic attached objects and dynamical notions.

M.Berger, *Géométrie vivante ou l'échelle de Jacob*, Cassini, 2009.

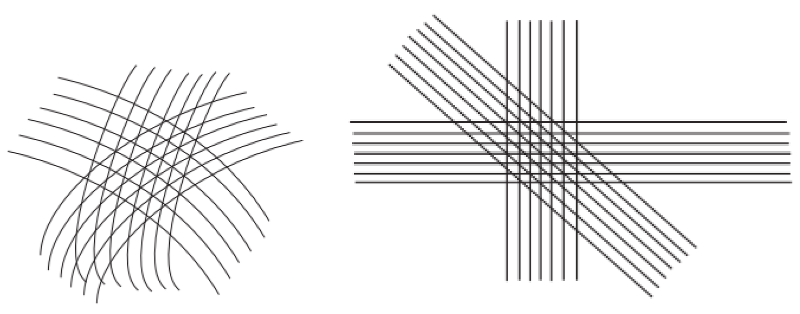
An heuristic approach to dynamical geometry

The basic idea behind **dynamical geometry** is that the geometrical properties of an object can be made transparent by looking how a given flow is modified by the existence of the object. Examples of such a relationship exist in nature as for example the Von Karman vortex streets which follow from the modification of the flow by an island.
This idea can be formalized as follows: A flow is mathematically encoded by a differential equation or a vector field. Let $\mathcal{M}$ be a given variety. We denote by $D(\mathcal{M})$ the set of differential equations leaving $\mathcal{M}$ **invariant**. An equivalent way to defined it is $D(\mathcal{M}) = \{\mathbf{v} \mid \mathbf{v}.F = K \times F\}$ where $\mathcal{M} = \{\mathbf{x} \in \mathbb{C}^n | F(\mathbf{x}) = 0\}$ and $\mathbf{v}$ are vector fields. The previous set is called **modulus of derivations** associated to $\mathcal{M}$ and was defined by Saito [S] in the 70-80's in a purely algebraic way.
We illustrate our point of view on the **Blaschke conjecture** concerning web geometry.

[S] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 27 (1980), no. 2, 265–291.

Web geometry and the Blaschke conjecture

An ***d*-web** $\mathcal{W}(d)$ is a collection of d-foliations $F_1, ..., F_d$ of codimension 1 such that their tangent space are in general position.



A d-web can be viewed as a differential equation:

$$F(x, y, y') = a_0(x, y)(y')^d + ... + a_d(x, y) = 0$$

or equivalently, $F(x, y, y') = a_0(x, y)\prod_{i=1}^{d}(y' - p_i(x, y))$ and a natural object associated is the discriminant:

$$\Delta = a_0(x, y)^{2d-2} \prod_{1 \le i < j \le d} (p_i(x, y) - p_j(x, y))^2.$$

[H]A.Hénaut, *Infinitesimal symmetries for planar webs through meromorphic connections*, 2016.

In general, the leaves of the web are not straight lines, but in certain cases by a change of coordinates the leaves of the foliations can be transformed in parallel straight lines, we talk about **parallelisation** or in a pencil of straight lines and we talk about **linearisation**.

Using classical approach to symmetries of dynamical systems we rederived a characterisation of the symmetries of a web obtained independently by A. Henaut [H].

Lie symmetries and parallelization

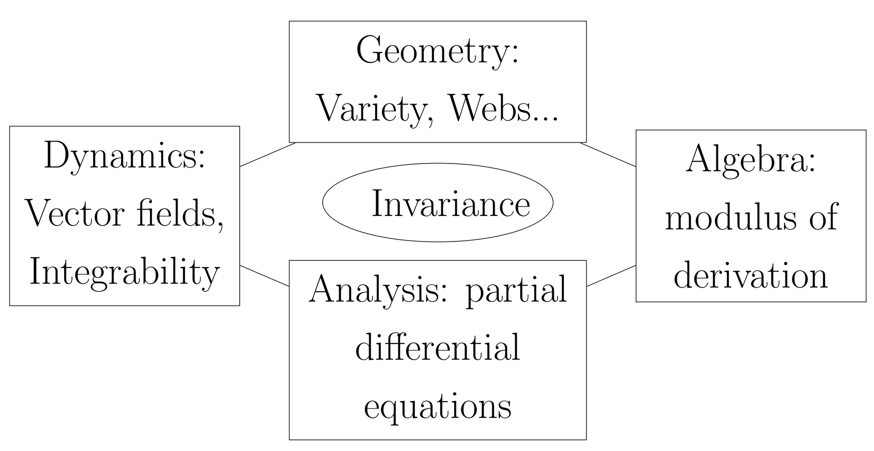
Let a 3−web $\mathcal{W}(3)$. If the dimension of its Lie algebra of symmetries is 3, then $\mathcal{W}(3)$ can be parallelized.

The problem of linearisation is known as the Blaschke conjecture. The Blashke conjecture was a priori solved following two different methods. These methods provide a computational way to verify if a web is linearisable or not. However, the two methods do not give the same answer. In particular for the web $F_1(x, y) = x$, $F_2(x, y) = y$ and $F(x, y) = (x + y)e^{-x}$, the Lyshiagin method states that it is not linearisable and the Munznay method states that it is. Using a characterization of linearisable web obtained by A. Henaut, we prove:

Theorem

The web defined by $F_1(x, y) = x$, $F_2(x, y) = y$ and $F(x, y) = (x + y)e^{-x}$ is not linearisable.

Link between webs and modulus of derivations: The symmetries of a web are contained in the modulus of derivation of Δ.



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